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THE ONSET OF THERMAL CONVECTION IN A FLUID LAYER WITH TIME-DEPENDENT TEMPERATURE DISTRIBUTION

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Abstract—The onset of convection in a fluid layer subject to internal heating and transient cooling from above is considered. The dynamic linear stability theory is applied and the dependency of the critical time on the Rayleigh number, Prandtl number, wave number of the perturbations, and the cooling rate is studied.

NOMENCLATURE

a	wave number
A_m	coefficients in equation (7)
b_n	coefficients in equation (10)
c	defined such that for $t > 0$, the upper surface temperature is lowered according to $\bar{T} = T_0 - ct$
$\mathbf{g}$	gravity vector
g	gravitational acceleration
L	depth of the fluid layer
Pr	Prandtl number
Ra	Rayleigh number defined in equation (12)
s	heat source
T	temperature
T_0	temperature at the lower surface
t	time
$\mathbf{u}$	velocity vector
u, v, w	perturbation components of velocity
x, y	coordinate axes in plane perpendicular to the gravity vector
z	coordinate axis in the direction of the gravity vector.

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
θ	perturbation component of temperature
λ	dummy variable in equation (11)
ν	kinematic viscosity
ρ	density.

Subscripts

1	in the x–y plane
c	critical value
i	initial
x, y	in x- or y-direction
RMS	root-mean-square.

Superscripts

$\bar{}$	horizontal average.

1. INTRODUCTION

THE ONSET of convection in a fluid layer subject to an unstable, time-independent nonlinear temperature profile has been studied by many investigators [1–4]. The approach used is that of linear stability theory which results in the critical Rayleigh numbers marking the neutral states. These results are independent of the Prandtl number. According to the principle of exchange of stability, these states of marginal instability separate the initial stationary quiescent states from those in which stationary convection patterns are established. Chandrasekhar [5] has given an extensive description of this method.

Foster [6, 7] considered the stability of an initially homogeneous layer of fluid which is suddenly cooled from above. Therefore, the mean temperature distribution was time dependent. He found that for a given Rayleigh number based on the cooling rate, the time needed for the system to manifest convective behavior, i.e. the critical time, depends on the Prandtl number of the fluid considered. He also found that for large Rayleigh numbers the critical time is independent of the depth of the fluid layer. Onat and Grigull [8] considered transient heating from below and in their semi-empirical study confirmed the dependency of the critical Rayleigh number on the Prandtl number. In a similar experiment, Davenport and King [9] also confirmed this Prandtl number dependency, and, since the fluid layers used in their experiments were relatively deep, they concluded that the time of the onset of convection is independent of the fluid depth. Mahler *et al.* [10] and Mahler and Schechter [11] performed experimental and analytical studies on the stability of a fluid layer where the change in density was due to gas absorption in the liquid. They used stochastically defined initial states in order to improve the agreement between the results of linear analysis and the experimental results. They concluded that the initial thermal molecular excitations do not affect the timing of the onset of convection. Davis and Choi [12] applied the same principle to the liquid film flow where downstream of the flow a linear and unstable temperature field was imposed. Their experimental and theoretical results were in agreement. Jhavari [13] examined the importance of the initial conditions by considering two cases. In one the instability was due to a step change in the surface temperature and the other was due to a ramp type change in the surface temperature. He concluded that a stochastic description of the thermodynamic fluctuations is the best choice for the initial condition. However, his results show that the results for the ramp type changes in the surface temperature are less dependent on the initial conditions. This is expected because as the buoyancy force slowly evolves it would be less influenced by the initial conditions.

In the stability analysis of fluid layers with time-independent stratifications the objective is to determine the critical Rayleigh numbers corresponding to no growth in the introduced disturbances. However, for time-dependent stratifications, an infinitesimal disturbance is introduced and its growth or decay is observed. If the buoyancy force is large enough to overcome the viscous and other forces, then the disturbances may grow in a rather superexponential manner. The determination of the time of the onset of convection is made empirically and is normally taken to be when the disturbance grows to a thousand times its initial magnitude, even though at this time the disturbances may no longer be infinitesimal and, consequently, the linearity assumption may no longer hold. It is the start of this superexponential growth which is important. As was mentioned above, by similar analysis Mahler and Schechter [11], Davis and Choi [12], and Foster [14] have found good agreement between the results of their prediction and the available experimental data.

In this study the onset of convection in a fluid layer with the following specifications is considered. Initially the fluid, which is confined between two shear-free surfaces, has been heated by a uniform source until steady-state temperature distribution is established. Later the fluid is cooled from above by lowering the temperature of the upper surface at a constant rate. Thus the fields are transient with the initial temperature distribution being nonlinear. The internal heat generation can be due to (a) absorption of external radiation by an optically thin layer [3, 4], (b) chemical reaction, and (c) radiation decay among others. The cooling from above can be due to evaporation [14, 15]. This fluid layer is assumed to be resting on another fluid with which it is immiscible. Therefore, the results of this analysis can be applied to fluid layers subjected to the above mentioned internal heating and upper surface evaporative cooling where the shear stresses at the boundaries are negligible.

The characteristic equation, obtained by manipulation of the continuity, momentum and thermal energy equations, is solved for a velocity perturbation which is assumed to be represented by a series with time-dependent coefficients. The resulting set of second-order ordinary differential equations is solved numerically.

2. ANALYSIS

2.1. *The characteristic equation*

Consider an initially quiescent horizontal Boussinesq fluid layer with a rectangular coordinate system in which gravity is in the z-direction and the origin is at the upper surface. In the absence of any mean motion the conservation of energy is written as

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \alpha\nabla^2 T + s, \tag{1}$$

where $\mathbf{u}$ is the perturbation velocity vector and s is the volumetric heat generation which is assumed not to be a function of the perturbation variables. By separating the temperature into horizontal mean and perturbation components, equation (1) for the horizontal mean component of temperature becomes

$$\frac{\partial \bar{T}}{\partial t} = \alpha\frac{\partial^2 \bar{T}}{\partial z^2} + s. \tag{2}$$

The equations expressing conservation of mass and momentum, assuming a linear equation of state, are written as

$$\nabla\cdot\mathbf{u} = 0, \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \beta\mathbf{g}\theta + \nu\nabla^2\mathbf{u}. \tag{4}$$

Following Pellew and Southwell [16], after linearization we have

$$\left[\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial}{\partial t} - \alpha\nabla^2\right)\nabla^2 - \beta g\frac{\partial \bar{T}}{\partial z}\nabla_1^2\right]w = 0. \tag{5}$$

For the velocity we assume a perturbation of the form

$$w(x, y, z, t) = w(z, t)\exp\left[\mathrm{i}(a_x x + a_y y)\right], \tag{6}$$

where $a^2 = a_x^2 + a_y^2$ is the horizontal wave number. A similar form is assumed for the temperature perturbation. It is assumed that the top and bottom surfaces are free and that $\theta = 0$ at $z = 0, L$.

It is further assumed that in equation (6) the first coefficient on the RHS can be separated and expressed in a series as

$$w(z, t) = \sum_{m=1}^{\infty} A_m(t)\sin\left(\frac{m\pi z}{L}\right), \tag{7}$$

which satisfies the boundary conditions. By applying equations (6) and (7) to equation (5), multiplying by $\sin(r\pi z/L)$ and integrating over $z = 0$ to L, one obtains

$$\begin{aligned} &\frac{L}{2}\left(\frac{r^2\pi^2}{L^2} + a^2\right)^3 \alpha\nu A_r - a^2\beta g\sum_{m=1}^{\infty} A_m \\ &\times\int_0^L \frac{\partial \bar{T}}{\partial z}\sin\left(\frac{m\pi z}{L}\right)\sin\left(\frac{r\pi z}{L}\right)\mathrm{d}z \\ &+\frac{L}{2}\left(\frac{r^2\pi^2}{L^2} + a^2\right)^2(\nu+\alpha)\frac{\mathrm{d}A_r}{\mathrm{d}t} \\ &+\frac{L}{2}\left(\frac{r^2\pi^2}{L^2} + a^2\right)\frac{\mathrm{d}^2A_r}{\mathrm{d}t^2} = 0, \quad r = 1, 2, \ldots \end{aligned} \tag{8}$$

Equation (8) consists of an infinite number of coupled second-order ordinary differential equations in $A_r(t)$. In order to solve these equations, the gradient of the mean temperature distribution must be specified. This time and space dependent gradient is obtained from equation (2).

2.2. *The mean temperature distribution*

The initial temperature distribution is due to uniform heat generation with both the upper and lower surfaces at a constant temperature, T_0. Therefore

$$\bar{T}_i = \frac{s}{2\alpha}(1-z)z + T_0. \tag{9}$$

With a half-range Fourier series, this can be written in a sine series as

$$\bar{T}_i = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi z}{L}\right) + T_0, \tag{10}$$

where

$$b_n = \frac{2}{L}\int_0^L \bar{T}_i(z)\sin\left(\frac{n\pi z}{L}\right)dz = \frac{[1-(-1)^n]2sL^2}{n^3\pi^3\alpha}.$$

For $t > 0$, the lower surface temperature is maintained at T_0 while the upper surface temperature is lowered at a constant rate of $-c$. The solution to this transient problem is found by separation of the variables and the application of Duhamel's theorem. The solution, given in Carslaw and Jaeger [17], is

$$\bar{T} - T_0 = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) e^{-\alpha n^2\pi^2 t/L^2} \times \left[\frac{Lb_n}{2} + \frac{n\alpha\pi}{L}\int_0^t e^{\alpha n^2\pi^2\lambda/L^2}(-c\lambda)\,d\lambda\right]. \tag{11}$$

Next the governing equations are made non-dimensional by defining

$$\bar{T}^* = \frac{\alpha(\bar{T}-T_0)}{sL^2},\quad z^* = \frac{z}{L},\quad t^* = \frac{\alpha t}{L^2},\quad w^* = \frac{wL}{\alpha},$$

$$a^* = aL,\quad \theta^* = \frac{\theta L^3 g\beta}{\nu\alpha},\quad Ra^* = \frac{g\beta s L^5}{\nu\alpha^2}, \tag{12}$$

$$A^* = \frac{AL}{\alpha},\quad b_n^* = \frac{\alpha b_n}{2sL^2},\quad c^* = \frac{c}{s}.$$

Note that the Rayleigh number is based on the initial uniform heating rate and that parameter c indicates the transient cooling. Equation (8), after dropping the asterisk, becomes

$$\frac{d^2A_r}{dt^2} = \frac{2a^2\,Ra\,Pr}{r^2\pi^2+a^2}\sum_{m=1}^{\infty} A_m(t)I_{mr}(t) - (r^2\pi^2+a^2)^2\,Pr\,A_r - (r^2\pi^2+a^2) \times (1+Pr)\frac{dA_r}{dt},\quad r = 1,2,\ldots, \tag{13}$$

where

$$I_{mr}(t) = \int_0^1 \frac{\partial\bar{T}}{\partial z}\sin(m\pi z)\sin(r\pi z)\,dz. \tag{14}$$

This can be written in the following convenient form

$$I_{mr} = \begin{cases} \dfrac{1}{2\pi^2}\left\{\dfrac{[1-(-1)^{m-r}-c]\,e^{-(m-r)^2\pi^2 t}+c}{(m-r)^2} - \dfrac{[1-(-1)^{m+r}-c]\,e^{-(m+r)^2\pi^2 t}+c}{(m+r)^2}\right\}, & \text{for } m \neq r, \\[2ex] \dfrac{ct}{2} + c\,\dfrac{[e^{-(m+r)^2\pi^2 t}-1]}{2\pi^2(m+r)^2}, & \text{for } m = r, \end{cases} \tag{15}$$

where, because the Fourier series in equation (11) is not uniformly convergent at $z = 0$, a closed form substitution was made [18]. The characteristic equation, i.e. equation (13), together with the contribution of the temperature gradient given by equation (15), can now be solved. The result is the transient behavior of the velocity perturbation.

3. SOLUTION

The infinite set of coupled second-order ordinary differential equations in $A_r(t)$ given by equation (13) are solved numerically. The Fourier series describing the velocity has been truncated to a finite number of terms. The scheme of the solution is Runge–Kutta–Gill's fourth-order approximation described by Romanelli [19]. The sensitivity of the solution to the initial conditions for A_r and dA_r/dt, as well as the number of terms retained in the Fourier series, is described in refs. [6, 10, 11]. Here the first 17 terms in the Fourier series were retained and the value of unity was assigned for the initial values of all A_r and the value of zero was assigned for the initial values of all dA_r/dt. From the results reported in refs. [6, 7, 10, 11] these choices seem to be reasonable for the ramp type change in the surface temperature considered here. The growth of the perturbation velocity can be presented by defining the flowing quantities

$$\bar{w}(t) = \left(\frac{\int_0^1 w(z,t)^2\,dz}{\int_0^1 w(z,t=0)^2\,dz}\right)^{1/2}, \tag{16}$$

$$w_{\text{RMS}}(z,t) = \left(\frac{w(z,t)^2\,dz}{\int_0^1 w(z,t=0)^2\,dz}\right)^{1/2}. \tag{17}$$

For validation of the solution scheme explained above, a case of uniform initial temperature was computed and the results were compared to those of Foster [6]. The two results were identical. In the numerical integration, as the Rayleigh number decreased, the required number of time steps increased. For example, 3740 time steps were required before $w(t)$ reached a magnitude of one

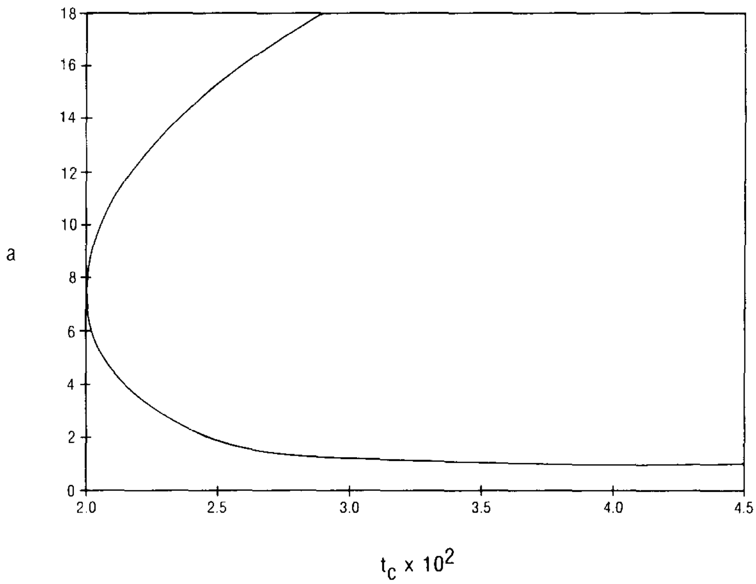

FIG. 1. Variation of the critical time with wave number: $Ra = 10^4$, $Pr = 7$ and $c = 10^3$.

thousand, for $Ra = 10^3$, $c = 10$ and a time-step size of 1.25×10^{-4}.

4. RESULTS AND DISCUSSION

The initial temperature distribution, i.e. the parabolic distribution, is potentially unstable. The critical Rayleigh number marking the state of marginal stability for this distribution is found by setting the time derivatives in equation (13) at zero. Then the critical Rayleigh number is found by requiring that the determinant of the infinite set of algebraic equations be equal to zero in order for A_m's to have non-trivial solutions. In practice, only a determinant of $O(10)$ needs to be considered. The minimum Rayleigh number with respect to wave number is sought. This value is also found by Kulacki and Goldstein [1] and is $Ra_c = 16\,992$, with a critical wave number of $a_c = 3.02$. In the study of the time-dependent profiles that follows, Rayleigh numbers below this critical value are considered.

The solution to equation (13) gives the transient behavior of the amplitude of velocity perturbation introduced in the potentially unstably stratified fluid layer considered here. Figures 1–4 show the dependency of the critical time, i.e. the time at which the vertically averaged velocity reaches a magnitude equal

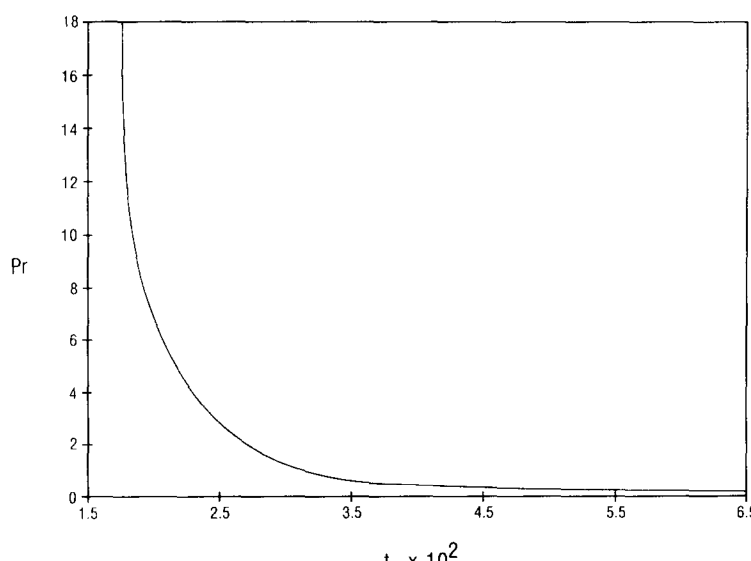

FIG. 2. Variation of the critical time with Prandtl number: $Ra = 10^4$, $c = 10^3$.

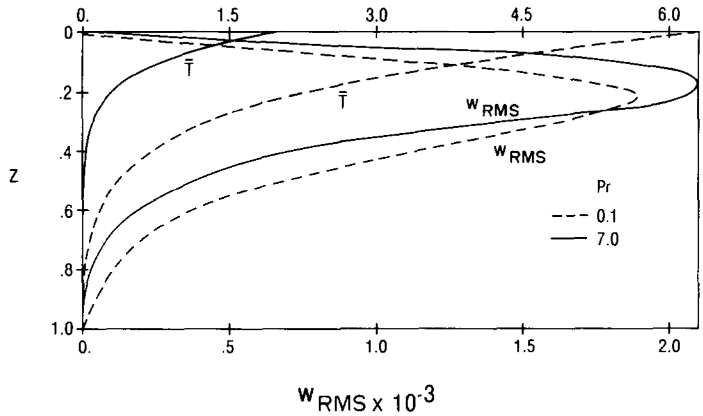

FIG. 3. Distributions of the mean temperature and the RMS of the vertical disturbance velocity. The solid curves are for $Pr = 7$ and the dashed curves are for $Pr = 0.1$: $Ra = 10^4$, $c = 10^3$.

to one thousand times its initial value, on various parameters of the problem.

Figure 1 shows the variation of the critical time with wave number. The results are for $Ra = 10^4$, $Pr = 7$, and $c = 10^3$. The wave number corresponding to the smallest critical time is called the critical wave number. The critical wave number increases with increases in Rayleigh number and cooling rate and in general depends on the Prandtl number. In all the results that follow, the wave number is the critical wave number.

Figure 2 shows the variation of the critical time with Prandtl number. The results are for $Ra = 10^4$ and $c = 10^3$. The trend is identical to that obtained by Foster [6] and Mahler *et al.* [10]. The critical time increases with decrease in Prandtl number. This indicates that as the thermal diffusivity increases the effect of cooling penetrates further into the fluid and, therefore, its effectiveness decreases. The results also show that for small thermal diffusivity, i.e. $Pr > 10$, the effect of cooling is so concentrated near the surface that the critical time which is small in magnitude does not vary significantly with the Prandtl number as long as $Pr > 10$. This Prandtl number dependency is also shown in Fig. 3. The results are for $Ra = 10^4$ and $c = 10^3$. The mean temperature distribution and the RMS of the vertical velocity disturbance at the critical time are shown. The solid curves show the results for $Pr = 7$ and dashed curves show the results for $Pr = 0.1$. Note that the location of the maximum w_{RMS} is at $z = 0.17$ for $Pr = 7$ and below that at $z = 0.22$ for $Pr = 0.1$. The

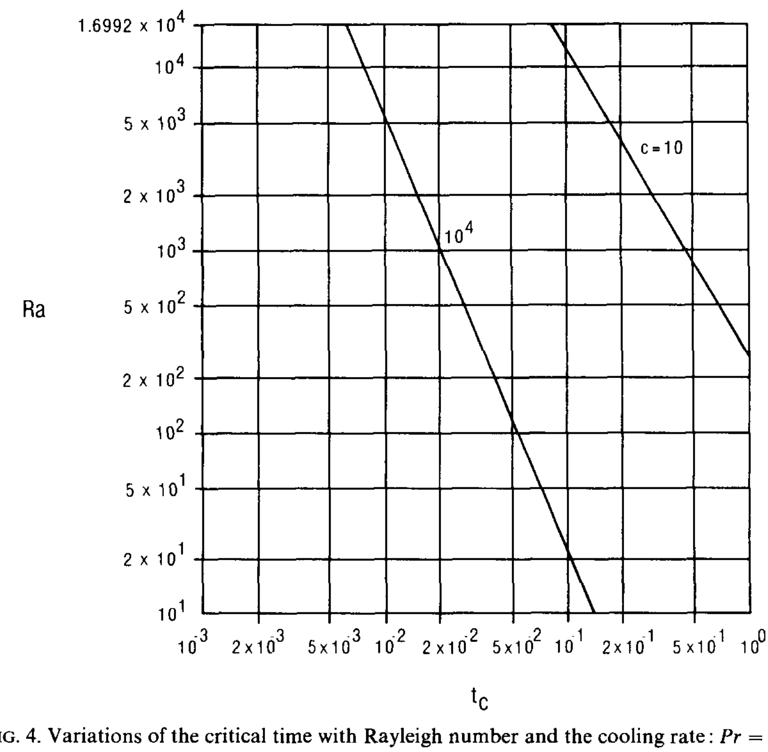

FIG. 4. Variations of the critical time with Rayleigh number and the cooling rate: $Pr = 7$.

surface temperature for $Pr = 0.1$ is larger than that for $Pr = 7$ because of the larger critical time associated with $Pr = 0.1$.

Figure 4 shows the variation of the critical time with respect to the Rayleigh number for two cooling rates and $Pr = 7$. Above $Ra = 16\,992$ the initial parabolic temperature distribution is unstable. Below $Ra = 16\,992$, the cooling of the upper surface is required for the onset of convection. The instability mechanism would therefore change slightly. For $Ra \ll 16\,992$, the initial parabolic temperature no longer plays a role and the results are similar to those for uniform initial temperature distribution. Therefore, the magnitude of the cooling rate is the dominating parameter. However, note that the cooling rate is non-dimensionalized with respect to the heating rate.

5. SUMMARY

Linear stability theory is applied to a fluid layer bounded by free surfaces and subject to a time-dependent temperature profile characterized by uniform initial heating and transient upper surface cooling. The results show that the critical time, defined as the time when the introduced disturbances grow to one thousand times their initial magnitude, increases as the Prandtl number decreases. The dependency of the critical time on the magnitude of the initial heating, i.e. Rayleigh number, and the magnitude of the cooling rate are quantitatively established.

Further investigations applying finite amplitude disturbances and rigid boundaries are recommended.

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REFERENCES

1. F. A. Kulacki and R. J. Goldstein, Hydrodynamic instability in fluid layers with uniform volumetric energy source, *Appl. Scient. Res.* **31**, 81–109 (1975).
2. E. M. Sparrow, R. J. Goldstein and V. K. Jonsson, Thermal instability in a horizontal layer: effect of boundary conditions and non-linear temperature profile, *J. Fluid Mech.* **18**, 513–528 (1964).
3. A. Yucel and Y. Bayazitoglu, Onset of convection in fluid layers with non-uniform volumetric energy source, *J. Heat Transfer* **101**, 606–671 (1979).
4. M. Tveitereid, Thermal convection in a horizontal fluid layer with internal heat sources, *Int. J. Heat Mass Transfer* **2**, 335–339 (1978).
5. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, pp. 428–436. Clarendon Press, Oxford (1961).
6. T. D. Foster, Stability of a homogeneous fluid cooled uniformly from above, *Physics Fluids* **8** (7), 1249–1257 (1965).
7. T. D. Foster, Effect of boundary conditions on the onset of convection, *Physics Fluids* **11** (6), 1257–1262 (1968).
8. K. Onat and U. Grigull, The onset of convection in a horizontal fluid layer heated from below, *Wärme- und Stoffübertragung* **3**, 103–113 (1970).
9. I. F. Davenport and C. J. King, The onset of natural convection from time-dependent profiles, *Int. J. Heat Mass Transfer* **17**, 69–76 (1974).
10. E. G. Mahler, R. S. Schechter and E. H. Wissler, Stability of a fluid layer with time-dependent density gradients, *Physics Fluids* **11** (9), 1901–1912 (1968).
11. E. G. Mahler and R. S. Schechter, The stability of a fluid layer with gas absorption, *Chem. Engng Sci.* **25**, 955–968 (1970).
12. E. J. Davis and C. K. Choi, Cellular convection with liquid-film flow, *J. Fluid Mech.* **81**, 565–592 (1977).
13. B. S. Jhavari, Rayleigh–Bénard convection from thermodynamic fluctuations, Ph.D. thesis, Stanford University, Stanford, California (1979).
14. T. D. Foster, Onset of manifest convection in a layer of fluid with a time-dependent surface temperature, *Physics Fluids* **12**, 2484–2487 (1969).
15. W. G. Spagenberg and W. R. Rowland, Convective circulation in water induced by evaporative cooling, *Physics Fluids* **4**, 743–750 (1961).
16. A. Pellew and R. V. Southwell, *Proc. R. Soc.* **A176**, 312–343 (1940).
17. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd edn.), pp. 102–104. Clarendon Press, Oxford (1959).
18. M. H. Ozisik, *Heat Conduction*, pp. 202–203. Wiley, New York (1980).
19. M. J. Romanelli, *Mathematical Methods for Digital Computer* (edited by A. Ralston and H. S. Wilf), pp. 110–120. Wiley, New York (1960).

APPARITION DE LA CONVECTION THERMIQUE DANS UNE COUCHE FLUIDE AVEC UNE DISTRIBUTION DE TEMPERATURE VARIABLE DANS LE TEMPS

Résumé—On considère l'apparition de la convection dans une couche fluide soumise à un chauffage interne et à un refroidissement transitoire sur le dessus. La théorie dynamique linéaire de stabilité est appliquée et on étudie la dépendance du temps critique vis-à-vis du nombre de Rayleigh, du nombre de Prandtl, du nombre d'onde des perturbations et de l'intensité du refroidissement.

DAS EINSETZEN DER THERMISCHEN KONVEKTION IN EINER FLUIDSCHICHT BEI ZEITABHÄNGIGER TEMPERATURVERTEILUNG

Zusammenfassung—Es wird das Einsetzen der Konvektion in einer Fluidschicht auf Grund von inneren Wärmequellen und zeitlich veränderlicher Kühlung von oben untersucht. Dazu wird die lineare Stabilitätstheorie angewendet und die Abhängigkeit der kritischen Zeit auf die Rayleigh-Zahl, die Prandtl-Zahl, die Wellenzahl der Störungen und die Kühlrate betrachtet.

ВОЗНИКНОВЕНИЕ ТЕПЛОВОЙ КОНВЕКЦИИ В СЛОЕ ЖИДКОСТИ С ЗАВИСИМЫМ ОТ ВРЕМЕНИ РАСПРЕДЕЛЕНИЕМ ТЕМПЕРАТУРЫ

Аннотация—Рассматривается возникновение конвекции в слое жидкости с внутренними источниками тепла при нестационарном охлаждении сверху. Используется линейная теория динамической устойчивости и исследуется зависимость критического времени от чисел Рэлея, Прандтля, волнового числа возмущений и скорости охлаждения.